

Smooth solutions for motion of a rigid body of general form in an incompressible perfect fluid

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Abstract

In this paper, we consider the interactions between a rigid body of general form and the incompressible perfect fluid surrounding it. Local well-posedness in the space $C([0, T]; H_s)$ is obtained for the fluid-rigid body system.

Keywords: Euler equations; Fluid-rigid body interaction; Exterior domain; Classical solutions

Mathematics Subject Classification(2000): 35Q30; 35Q35

1 Introduction

In this paper, we investigate the motion of a solid in an incompressible perfect fluid. The behavior of the fluid is described by the Euler equations. The solid we consider is a rigid body, whose movement consists of translation and rotation. And its motion conforms to the Newton's law. Assume that both the fluid and the rigid body are homogeneous. For simplicity of writing, the density of the fluid equals to 1. The domain occupied by the solid at the time is $\mathcal{O}(t)$, and $\Omega(t) = \mathbb{R}^3 \setminus \overline{\mathcal{O}(t)}$ is the domain occupied by the fluid. Suppose

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$\mathcal{O}(0) = \mathcal{O}$ and $\Omega(0) = \Omega$ share a smooth boundary $\partial\mathcal{O}$ (or $\partial\Omega$). The equations modeling the dynamics of the system read(see also [18])

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = f, \quad \text{in } \Omega(t) \times [0, T], \quad (1.1)$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega(t) \times [0, T], \quad (1.2)$$

$$u \cdot \vec{n} = (h' + \omega \times (x - h(t))) \cdot \vec{n}, \quad \text{on } \partial\Omega(t) \times [0, T], \quad (1.3)$$

$$\lim_{|x| \rightarrow \infty} u(x, t) = u_\infty, \quad (1.4)$$

$$mh'' = \int_{\partial\Omega(t)} p \vec{n} d\sigma + f_{rb}, \quad \text{in } [0, T], \quad (1.5)$$

$$(J\omega)' = \int_{\partial\Omega(t)} (x - h(t)) \times p \vec{n} d\sigma + T_{rb}, \quad \text{in } [0, T], \quad (1.6)$$

$$u(x, 0) = u_0(x) \quad x \in \Omega, \quad (1.7)$$

$$h(0) = 0 \in \mathbb{R}^3, \quad h'(0) = l_0 \in \mathbb{R}^3, \quad \omega(0) = \omega_0 \in \mathbb{R}^3. \quad (1.8)$$

In the above system, u and p are the velocity field and the pressure of the fluid respectively. f is the external force field applied to the fluid. f_{rb} and T_{rb} denote the external force and the external torque for the rigid body respectively. m is the mass, and J is the inertia matrix moment related to the mass center of the solid. Suppose the density of the rigid body is ρ , then

$$m = \int_{\mathcal{O}(t)} \rho dx = \int_{\mathcal{O}} \rho dx,$$

and

$$[J(t)]_{kl} = \int_{\bar{\mathcal{O}}(t)} \rho [|x - h(t)|^2 \delta_{kl} - (x - h(t))_k (x - h(t))_l] dx.$$

Here $h(t)$ denotes the position of the mass center of the rigid body and δ_{kl} is the Kronecker symbol. And denote $J(0)$ by \bar{J} . $\omega(t)$ is the angular velocity of the rigid body. \vec{n} is the unit outward normal to $\partial\Omega(t)$.

Assume that the center of \mathcal{O} is the origin, i.e.,

$$\int_{\mathcal{O}} y dy = 0 \in \mathbb{R}^3.$$

Let $Q(t)$ be a rotation matrix associated with the angular velocity $\omega(t)$ of the rigid body, which is the solution of the following initial value problem:

$$\begin{cases} \frac{dQ(t)}{dt} = A(\omega(t))Q(t) \\ Q(0) = Id. \end{cases} \quad (1.9)$$

Here

$$A(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

and Id is the identity matrix. Then the domain $\mathcal{O}(t)$ is given by

$$\mathcal{O}(t) = \{Q(t)y + h(t) : y \in \mathcal{O}\}.$$

For simplicity, we assume that $f = 0, u_\infty = 0, f_{rb} = 0$ and $T_{rb} = 0$.

For the case that the fluid is viscous, there have been many results over the last two decades. The existence of global weak solutions of the above system was proved by [4, 5, 8, 9, 10, 13, 19, 20]. If the rigid body is a disk in \mathbb{R}^2 , T.Takahashi and M.Tucsnak[23] showed the existence and uniqueness of global strong solutions. Later, P.Cusmille and T.Takahashi [2] extended the result to general rigid body case in \mathbb{R}^2 . They also proved the local existence and uniqueness of strong solutions in \mathbb{R}^3 .

For the case that the fluid is inviscid, [17] dealt with this problem first. When the solid is of C^1 and piecewise C^2 boundary, and the fluid fills in \mathbb{R}^2 , a unique global classical solution was obtained under some assumption on the initial vorticity in [17]. A global weak solution was constructed in [25] when the initial data belongs to $W^{1,p}$, $p > \frac{4}{3}$. Recently, C. Rosier and L. Rosier[18] proved the local existence of $W^{s,2}$ -strong solutions for $d \geq 2$, $s \geq [d/2] + 2$ and the solid is a ball. The key idea is to make use of the Kato-Lai theory, which was originated from [14].

At the same time, the fluid-rigid body system which occupies a bounded domain was also studied. In particular, [11] proved the existence and uniqueness of strong solution for the three dimensional case. The approach of [11] follows closely the idea in [1], which is used to study the classical Euler equations. Their method also applies to the case that there are several solids in the fluid.

In this paper, we plan to extend the result of [18] to a more general setting. We will study the case that the solid is smooth and of a general form. As shown by the system itself, it is a free boundary problem. To deal with the free boundary problem, usually the first step is to fix the boundary. To fulfill that, [18] made a translation of the coordinate system. However, this special transformation can only be applied to the case that the solid is a ball. The solid in this paper is of general form, hence a different change of coordinates from [18] should be introduced. As said before, the motion of the solid is a rigid body movement, so a natural idea is to use a coordinate transformation consisting of the translation and rotation of the solid. We tried in this way. As will be shown in section 2, after the transformation, the new equivalent system has some term difficult to control. So we gave up the idea and then applied a new transformation which coincides

with the movement of the solid in its neighborhood and becomes identity when far away from it. The concrete form of the transformation will be given in section 2. In fact, this kind of transformation has been used by [12, 2, 6].

For the new equivalent system after the transformation, we will use the Kato-Lai theory[14] to construct a sequence of approximate solutions and prove the limit is the solution required. The Kato-Lai theory is a Galerkin method in spirit. One can construct the Galerkin approximation by himself or herself. We use the theory here directly to avoid unnecessary details. By the way, the method we apply here can also be used to deal with the several-solids case after some minor modification.

2 Transformed equations and main result

As noted above, to fix the boundary, one method is to use the coordinates transformation. As shown in [16, 17], a direct way is the following one consisting of translation and rotation,

$$\begin{aligned} x &= Q(t)y + h(t), & \bar{u}(y, t) &= Q(t)^T u(Q(t)y + h(t), t), \\ \bar{p}(y, t) &= p(Q(t)y + h(t), t), & \bar{h}(t) &= \int_0^t Q(s)^T h'(s) ds, \\ \bar{J} &= J(0), & \bar{\omega}(y, t) &= Q(t)^T \omega, \end{aligned}$$

where $Q(t)$ is given in section 1 and $Q(t)^T$ is the transpose of $Q(t)$.

After the transformation, an equivalent system is obtained as follows:

$$\frac{\partial \bar{u}}{\partial t} + [(\bar{u} - \bar{h}' - \bar{\omega} \times y) \cdot \nabla] \bar{u} + \bar{\omega} \times \bar{u} + \nabla \bar{p} = 0, \quad \text{in } \Omega \times [0, T], \quad (2.1)$$

$$\operatorname{div} \bar{u} = 0, \quad \text{in } \Omega \times [0, T], \quad (2.2)$$

$$\bar{u} \cdot \vec{n} = (\bar{h}' + \bar{\omega} \times y) \cdot \vec{n}, \quad \text{on } \partial\Omega \times [0, T], \quad (2.3)$$

$$m\bar{h}'' = \int_{\partial\Omega} \bar{p}\vec{n}d\sigma - m\bar{\omega}(t) \times \bar{h}'(t), \quad \text{in } [0, T], \quad (2.4)$$

$$\bar{J}\bar{\omega}' = \int_{\partial\Omega} y \times \bar{p}\vec{n}d\sigma + (\bar{J}\bar{\omega}(t)) \times \bar{\omega}(t), \quad \text{in } [0, T], \quad (2.5)$$

$$\bar{u}(y, 0) = u_0, \quad y \in \Omega, \quad (2.6)$$

$$\bar{h}(0) = 0, \quad \bar{h}'(0) = l_0, \quad \bar{\omega}(0) = \omega_0. \quad (2.7)$$

The new problem is a fixed boundary problem now. However, there is a term $[\bar{\omega} \times y) \cdot \nabla] \bar{u}$, whose coefficient become unbounded at large spatial distance. For the 2D case, the difficulty was overcome in [17] by assuming that u_0 belongs to some weighted space. The method there depends strongly on the fact that the vorticity satisfies a transport equation in 2D. However, the fact does not hold any more in 3D. To avoid this difficulty, we will

use another change of variables. The new transformation coincides with $Q(t)y + h(t)$ in a neighborhood of the solid and becomes identity when far away from it. In fact, this transformation was initially applied by Inoue and Wakimoto [12], later by Dintelmann etc.[6] and Cusmille and Takahashi [2]

More precisely, for a pair of continuous vector-valued functions $(l(t), \omega(t))$, let

$$h(t) = \int_0^t l(s)ds, \quad t \in [0, T], \quad (2.8)$$

and

$$V(x, t) = l(t) + \omega(t) \times (x - h(t)), \quad (x, t) \in \mathbb{R}^3 \times [0, T], \quad (2.9)$$

which is a rigid body movement.

Choose a smooth function $\xi : \mathbb{R}^3 \rightarrow \mathbb{R}$ with compact support such that $\xi(y) = 1$ in a neighborhood of $\bar{\mathcal{O}}$, and set

$$\psi(x, t) = \xi(Q(t)^T(x - h(t))),$$

where $Q(t)$ is given by (1.9).

Then introduce the functions W and Λ ,

$$W(x, t) = \frac{1}{2}l(t) \times (x - h(t)) + \frac{|x - h(t)|^2}{2}\omega, \quad (2.10)$$

$$\Lambda(x, t) = \psi V + \nabla \psi \times W. \quad (2.11)$$

It is easy to check that Λ satisfies the following two lemmas(or refer to [2, 6]).

Remark 2.1. *In what follows you will see the function Λ will produce a transformation which coincides with $Q(t)y + h(t)$ in the neighborhood of the solid and becomes identity when far away from it. The reason why we use Λ instead of ψV is that we need the function to be of divergence free. The function W is introduced to eliminate the divergence of ψV .*

Lemma 2.2. (1) $\Lambda(x, t) = 0$, if x is far away from $\mathcal{O}(t)$;

(2) $\Lambda(x, t) = l(t) + \omega(t) \times (x - h(t))$ in $\mathcal{O}(t) \times [0, T]$;

(3) $\operatorname{div} \Lambda = 0$ in $\mathbb{R}^3 \times [0, T]$;

(4) For all $t \in [0, T]$, $\Lambda(\cdot, t)$ is a $C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ function. Moreover, for every $s \in \mathbb{N}$,
 $\|\Lambda(\cdot, t)\|_{W^{s, \infty}(\mathbb{R}^3)} \leq C(s, T)(|l(t)| + |\omega(t)|)$;

(5) For all $x \in \mathbb{R}^3$, the function $\Lambda(x, \cdot)$ is in $C^0([0, T]; \mathbb{R}^3)$, provided that $l, \omega \in C^0[0, T]$.

Next, consider the vector field $X(y, t)$ which satisfies

$$\begin{cases} \frac{\partial X(y, t)}{\partial t} = \Lambda(X(y, t), t), & t \in (0, T], \\ X(y, 0) = y \in \mathbb{R}^3. \end{cases} \quad (2.12)$$

Lemma 2.3. *For every $y \in \mathbb{R}^3$, the initial-value problem (2.12) admits a unique solution $X(y, \cdot) : [0, T] \rightarrow \mathbb{R}^3$, which is a C^1 function on $[0, T]$. Moreover, the solution has the following properties:*

(1) *For all $t \in [0, T]$, the mapping $y \mapsto X(y, t)$ is a C^∞ diffeomorphism from \mathcal{O} onto $\mathcal{O}(t)$ and from Ω onto $\Omega(t)$.*

(2) *Denote by $Y(\cdot, t)$ the inverse of $X(\cdot, t)$. Then for every $x \in \mathbb{R}^3$ the mapping $t \mapsto Y(x, t)$ is C^1 -continuous and satisfies the following initial value problem,*

$$\begin{cases} \frac{\partial Y(x, t)}{\partial t} = - \left[\frac{\partial X(Y(x, t), t)}{\partial y} \right]^{-1} \Lambda(x, t), & t \in (0, T], \\ Y(x, 0) = x \in \mathbb{R}^3. \end{cases} \quad (2.13)$$

(3) *For every $x, y \in \mathbb{R}^3$ and for every $t \in [0, T]$, the determinants of the Jacobian matrices J_X of $X(y, t)$ and J_Y of $Y(x, t)$ both equal to 1, i.e.,*

$$\det(J_X(y, t)) = \det(J_Y(x, t)) = 1. \quad (2.14)$$

Let

$$\begin{aligned} x &= X(y, t), & v(y, t) &= J_Y(X(y, t), t)u(X(y, t), t), \\ q(y, t) &= p(X(y, t), t), & H(t) &= Q(t)^T h(t), \quad L(t) = Q(t)^T l(t), \end{aligned} \quad (2.15)$$

and $R(t)$ is the vector-valued function satisfying

$$A(R(t)) = Q(t)^T A(\omega(t)) Q(t). \quad (2.16)$$

Denote

$$\begin{cases} G(y, t) = (g^{ij}(y, t)) = \left(\sum_{k=1}^3 \frac{\partial Y_i}{\partial x_k}(X(y, t), t) \frac{\partial Y_j}{\partial x_k}(X(y, t), t) \right), \\ g_{ij} = \sum_{k=1}^3 \frac{\partial X_k}{\partial y_i}(y, t) \frac{\partial X_k}{\partial y_j}(y, t), \\ \Gamma_{i,j}^k(y, t) = \frac{1}{2} \sum_{l=1}^3 g^{kl} \left\{ \frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l} \right\}. \end{cases} \quad (2.17)$$

Now one can transform the original system (1.1)-(1.8) into the following system, which is a fixed boundary problem, (see [2, 15])

$$\frac{\partial v}{\partial t} + Mv + Nv + G \cdot \nabla q = 0, \quad \text{in } \Omega \times [0, T], \quad (2.18)$$

$$\operatorname{div} v = 0, \quad \text{in } \Omega \times [0, T], \quad (2.19)$$

$$v(y, t) \cdot \vec{n} = (L(t) + R(t) \times y) \cdot \vec{n}, \quad \text{on } \partial\Omega \times [0, T], \quad (2.20)$$

$$mL'(t) = \int_{\partial\Omega} q \vec{n} d\sigma - mR(t) \times L(t), \quad \text{in } [0, T], \quad (2.21)$$

$$\bar{J}R'(t) = \int_{\partial\Omega} y \times q \vec{n} d\sigma + \bar{J}R(t) \times R(t), \quad \text{in } [0, T], \quad (2.22)$$

$$v(y, 0) = u_0(y), \quad y \in \Omega, \quad (2.23)$$

$$H(0) = 0, \quad L(0) = l_0, \quad R(0) = \omega_0. \quad (2.24)$$

where

$$\begin{cases} (Mv)_i = \sum_{j=1}^3 \frac{\partial Y_j}{\partial t} \frac{\partial v_i}{\partial y_j} + \sum_{j,k=1}^3 \left\{ \Gamma_{j,k}^i \frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k} \frac{\partial^2 X_k}{\partial t \partial y_j} \right\} v_j; \\ (Nv)_i = \sum_{j=1}^3 v_j \frac{\partial v_i}{\partial y_j} + \sum_{j,k=1}^3 \Gamma_{j,k}^i v_j v_k; \\ (G \cdot \nabla q)_i = \sum_{j=1}^3 g^{ij} \frac{\partial q}{\partial y_j}. \end{cases} \quad (2.25)$$

Now the main result in this paper reads

Theorem 2.4. *Suppose that $s > \frac{5}{2}$, $u_0 \in H^s(\Omega)$ and $u_0 \cdot \vec{n} = (l_0 + \omega_0 \times y) \cdot \vec{n}$ on $\partial\Omega$. Then there exist some $T_0 > 0$ and a unique solution (v, q, L, R) of (2.18)-(2.24) such that*

$$v \in C([0, T_0]; H^s(\Omega)), \quad \nabla q \in C([0, T_0]; H^{s-1}(\Omega)),$$

and

$$L, R \in C^1([0, T_0]; \mathbb{R}^3).$$

Such a solution is unique up to an arbitrary function of t which may be added to q . Furthermore, T_0 does not depend on s .

Remark 2.5. *Following similar proof in section 7 for the uniqueness of the solution, we can get the stability of the solution. The method is standard as for the classical Euler equations, so we omit the details.*

Remark 2.6. *There are still many questions for the fluid-rigid body system. For example, whether the strong solution blows up or not? If there are many solids in the fluid, is there*

any collision between these solids? Which comes first, the collision or the blow up of $\|\nabla v\|_{L^\infty}$?

For readers' convenience, we list the outline of the following sections. Section 3 will be devoted to some notations or definitions for the function spaces and some lemmas to be used in the proof of the main result. The preliminaries contain the decomposition of the function spaces associated with this particular problem, the Kato-Lai theory for the construction of approximate solutions, and the estimates for the coordinate transformation. In section 4, we will give the a priori estimates for the new pressure term ∇q in (2.18), under the assumption that $\|v\|_{H^s(\Omega)\cap\tilde{X}}$ is bounded. These estimates will be used for the uniform estimates for the approximate solutions, which are constructed in section 5. Moreover, some uniform estimates for these solutions are derived at the same time. The convergence of these solutions is studied in section 6. And the limit is exactly the solution required. In the last section, section 7, we prove the uniqueness of the solution. Making use of the uniqueness result, the continuity of the solution with respect to time t in some space H_s is also proved.

3 Preliminaries

Before stating our main result, we'd like to introduce some notations and definitions.

Suppose \mathcal{S} is a domain in \mathbb{R}^3 . $L^2(\mathcal{S})$ is the space of L^2 -integrable functions with the standard inner product $(\cdot, \cdot)_{L^2(\mathcal{S})}$. By the way, we will not distinguish the scalar function spaces and the corresponding vector-valued function spaces.

Suppose s is a nonnegative integer, then

$$H^s(\mathcal{S}) = \{u \in L^2(\mathcal{S}) : D^\alpha u \in L^2(\mathcal{S}), \quad \forall \alpha \in \mathbb{N}^3, s.t., |\alpha| \leq s\},$$

with the inner product

$$(u, v)_{H^s(\mathcal{S})} = \sum_{|\alpha| \leq s} (D^\alpha u, D^\alpha v)_{L^2(\mathcal{S})}.$$

The homogeneous Sobolev space

$$D^{1,2}(\mathcal{S}) = \{u \in L^1_{loc}(\mathcal{S}) : \nabla u \in L^2(\mathcal{S})\},$$

with the seminorm

$$|u|_{D^{1,2}(\mathcal{S})} = \|\nabla u\|_{L^2(\mathcal{S})}.$$

If one identifies the two functions $u_1, u_2 \in D^{1,2}(\mathcal{S})$ whenever $|u_1 - u_2|_{D^{1,2}(\mathcal{S})} = 0$, i.e., u_1 and u_2 differ by a constant, the quotient space $\dot{D}^{1,2}(\mathcal{S})$, with the norm $|\cdot|_{D^{1,2}(\mathcal{S})}$, is

derived. In the following text, without any confusion, we do not distinguish the elements in $D^{1,2}(\mathcal{S})$ and $\dot{D}^{1,2}(\mathcal{S})$ very strictly.

Let $B_R(y)$ denote the ball centered at y and with the radius R . $\Omega_R := \Omega \cap B_R(0)$. Let $\rho = \frac{m}{|\mathcal{O}|}$, where $|\mathcal{O}|$ stands for the volume of \mathcal{O} . Hence ρ is the density of the solid. Let $\tilde{X} = L^2(\mathbb{R}^3)$ be endowed with the inner product,

$$(u, v)_{\tilde{X}} = \int_{\Omega} u(y) \cdot v(y) dy + \rho \int_{\mathcal{O}} u(y) \cdot v(y) dy.$$

Define

$$\tilde{X}_* = \{u \in \tilde{X} : \operatorname{div} u = 0 \text{ in } \mathbb{R}^3, \exists l, \omega \in \mathbb{R}^3, s.t., u = l + \omega \times y \text{ in } \mathcal{O}\},$$

which is a closed subspace of \tilde{X} .

Remark 3.1. For every $u \in \tilde{X}_*$, and suppose that $u = l + \omega \times y$ on \mathcal{O} . In fact, l and ω are unique vectors satisfying the above relation, and

$$l = \frac{1}{m} \int_{\mathcal{O}} \rho u dy, \quad \omega = -\bar{J}^{-1} \int_{\mathcal{O}} (u \times y) dy.$$

The fact has been proved, see [3] or [26]. In what follows, we will denote the vectors l, ω associated with $u \in \tilde{X}_*$ by l_u, ω_u .

Let $H_s = \{u \in \tilde{X} : u|_{\Omega} \in H^s(\Omega)\}$ be endowed with the scalar product

$$(u, v)_{H_s} = (u, v)_{H^s(\Omega)} + \rho(u, v)_{L^2(\mathcal{O})}.$$

V_s is the space of functions $v \in H_s$ such that $v|_{\Omega}$ belongs to $\mathcal{D}(A)$, where A is the elliptic operator $Af = \sum_{|\alpha| \leq s} (-1)^{|\alpha|} \partial^{2\alpha} f$ with Neumann boundary conditions, and $\mathcal{D}(A) \subseteq H^{2s}(\Omega)$. V_s is endowed with the scalar product

$$(u, v)_{V_s} = (u, v)_{H^{2s}(\Omega)} + \rho(u, v)_{L^2(\mathcal{O})}.$$

As in [18], we introduce a bilinear form on $V_s \times \tilde{X}$:

$$\langle v, u \rangle = \left(\sum_{|\alpha| \leq s} (-1)^{|\alpha|} \partial^{2\alpha} v, u \right)_{L^2(\Omega)} + \rho(v, u)_{L^2(\mathcal{O})}.$$

Since the main idea in this paper is the Kato-Lai theory, so we'd like to give a brief description of the theory, which is cited from [18]. For more details, please refer to [14]. Let V, H, X be three real separable Banach spaces. We say that $\{V, H, X\}$ is an admissible triplet if the following conditions hold.

- $V \subset H \subset X$, and the inclusions are dense and continuous.
- H is a Hilbert space, with inner product $(\cdot, \cdot)_H$ and norm $\|\cdot\|_H = (\cdot, \cdot)_H^{\frac{1}{2}}$.
- There is a continuous, nondegenerate bilinear form on $V \times X$, denoted by $\langle \cdot, \cdot \rangle$, such that

$$\langle v, u \rangle = (v, u)_H \quad \text{for all } v \in V, u \in H.$$

Recall that the bilinear form $\langle v, u \rangle$ is continuous and nondegenerate when

$$|\langle v, u \rangle| \leq C \|v\|_V \|u\|_X \quad \text{for some constant } C > 0; \quad (3.1)$$

$$\langle v, u \rangle = 0 \quad \text{for all } u \in X \text{ implies } v = 0; \quad (3.2)$$

$$\langle v, u \rangle = 0 \quad \text{for all } v \in V \text{ implies } u = 0. \quad (3.3)$$

A map $A : [0, T] \times H \rightarrow X$ is said to be sequentially weakly continuous if $A(t_n, v_n) \rightharpoonup A(t, v)$ in X whenever $t_n \rightarrow t$ and $v_n \rightharpoonup v$ in H . We denote by $C_w([0, T]; H)$ the space of sequentially weakly continuous functions from $[0, T]$ to H , and by $C_w^1([0, T]; X)$ the space of the functions $u \in W^{1, \infty}(0, T; X)$ such that $\frac{du}{dt} \in C_w([0, T], X)$.

We are concerned with the Cauchy problem

$$\frac{dv}{dt} + A(t, v) = 0, \quad \text{with } v(0) = v_0. \quad (3.4)$$

The Kato-Lai existence result for abstract evolution equations is as follows.

Theorem 3.2. *Let $\{V, H, X\}$ be an admissible triplet. Let A be a sequentially weakly continuous map from $[0, T] \times H$ into X such that*

$$\langle v, A(t, v) \rangle \geq -\beta(\|v\|_H^2) \quad \text{for } t \in [0, T], v \in V,$$

where $\beta(r) \geq 0$ is a continuous nondecreasing function of $r \geq 0$. For any $v_0 \in H$, consider the ODE $\gamma'(t) = \beta(\gamma(t))$, $\gamma(0) = \|v_0\|_H^2$. Suppose the maximal time of existence of γ is T_0 , then for (3.4), there exists a solution v of (3.4) in the class

$$v \in C_w([0, T_0]; H) \cap C_w^1([0, T_0]; X).$$

Moreover, one has

$$\|v(t)\|_H^2 \leq \gamma(t), \quad t \in [0, T_0],$$

In fact, it was proved in [18] that the triplet $\{\tilde{X}, H_s, V_s\}$ is admissible for any smooth boundary.

The following lemma gives a decomposition of $L^2(\mathbb{R}^3)$. In particular, the second part is a replacement of Proposition 3.1 of [18].

Lemma 3.3. Let $G_1^2 = \{u \in L^2(\mathbb{R}^3) : u = \nabla q_1, q_1 \in L_{\text{loc}}^1(\mathbb{R}^3)\}$, and

$$G_2^2 = \left\{ u \in L^2(\mathbb{R}^3) : \begin{aligned} &\text{div } u = 0 \text{ in } \mathbb{R}^3, \quad u = \nabla q_2 \text{ in } \Omega, \quad q_2 \in L_{\text{loc}}^1(\Omega), \quad u = \phi \text{ in } \mathcal{O}, \\ &\int_{\mathcal{O}} \phi dy = - \int_{\partial\Omega} q_2 \vec{n} \, d\sigma, \quad \text{and} \quad \int_{\mathcal{O}} \phi \times y dy = - \int_{\partial\Omega} q_2 \vec{n} \times y \, d\sigma \end{aligned} \right\}.$$

Then (1) \tilde{X}_*, G_1^2 and G_2^2 are mutually orthogonal in the sense of the standard L^2 -inner product and

$$L^2(\mathbb{R}^3) = \tilde{X}_* \oplus G_1^2 \oplus G_2^2.$$

It means that for every $u \in L^2(\mathbb{R}^3)$,

$$u(y) = \begin{cases} u_1 + \nabla q_1 + \nabla q_2, & y \in \Omega \\ u_1 + \nabla q_1 + \phi, & y \in \mathcal{O} \end{cases} \in \tilde{X}_* \oplus G_1^2 \oplus G_2^2. \quad (3.5)$$

Suppose $u_1 = l_{u_1} + \omega_{u_1} \times y$ in \mathcal{O} , then

$$|l_{u_1}| + |\omega_{u_1}| \leq C \|u_1\|_{L^2(\mathbb{R}^3)} \leq C \|u\|_{L^2(\mathbb{R}^3)} \leq C \|u\|_{\tilde{X}}. \quad (3.6)$$

(2) Denote the projector which maps $L^2(\mathbb{R}^3)$ to \tilde{X}_* by \mathbb{P} . In fact, \mathbb{P} maps H_s into H_s continuously for any $s \geq 0$.

Proof. The orthogonal decomposition in (1) has been proved in [3]. And the estimate (3.6) is partly derived in [3] and partly due to the fact that $\|\cdot\|_{L^2(\mathbb{R}^3)}$ and $\|\cdot\|_{\tilde{X}}$ are equivalent norms. Now we verify that (2) holds. For every $u \in H_s (s \geq 0)$, suppose that $u = u_1 + \nabla q_1 + \nabla q_2 = u_1 + \nabla P$ over Ω , then it suffices to prove that

$$\|\nabla P\|_{H^s(\Omega)} \leq C \|u\|_{H_s},$$

with some constant C independent of u .

In fact, from the formula (3.5), P satisfies the following equations:

$$\begin{cases} \Delta P = \text{div } u, & \text{in } \Omega, \\ \frac{\partial P}{\partial \vec{n}} = u \cdot \vec{n} - (l_{u_1} + \omega_{u_1} \times y) \cdot \vec{n}, & \text{on } \partial\Omega. \end{cases} \quad (3.7)$$

Let

$$\varphi = \nabla \times \left[\frac{1}{2} \xi(y) \cdot (l_{u_1} \times y - \omega_{u_1} |y|^2) \right],$$

where ξ is a cut-off function defined in section 2. Clearly, $\text{div } \varphi = 0$ in Ω and $\varphi \cdot \vec{n} = (l_{u_1} + \omega_{u_1} \times y) \cdot \vec{n}$ on $\partial\Omega$. Therefore, (3.7) can be rewritten,

$$\begin{cases} \Delta P = \text{div } (u - \varphi), & \text{in } \Omega, \\ \frac{\partial P}{\partial \vec{n}} = (u - \varphi) \cdot \vec{n}, & \text{on } \partial\Omega. \end{cases} \quad (3.8)$$

The solution to the system (3.8) is closely related to the Helmholtz-Weyl decomposition. As proved in [7],

$$\begin{aligned}
\|\nabla P\|_{H^s(\Omega)} &\leq C\|u - \varphi\|_{H^s(\Omega)} \\
&\leq C(\|u\|_{H^s(\Omega)} + \|\varphi\|_{H^s(\Omega)}) \\
&\leq C(\|u\|_{H^s(\Omega)} + |l_{u_1}| + |\omega_{u_1}|) \\
&\leq C\|u\|_{H^s},
\end{aligned}$$

which completes the proof of Lemma 3.3. \square

The following lemma is to give the bounds of the terms which appear in the system (2.18)-(2.24). Before that we'd like to give another description of relationship between (l, ω) and (L, R) , which is different from (2.15)-(2.16). Suppose (L, R) is given, we want to determine (l, ω) and then define the other coefficients in (2.18)-(2.24).

Since

$$\frac{dQ(t)}{dt} = A(\omega(t))Q(t),$$

multiplying by $Q^T(t)$, then

$$-\frac{dQ^T(t)}{dt}Q(t) = A(R(t)).$$

It gives that

$$\begin{cases} \frac{dQ(t)}{dt} = Q(t)A(R(t)), \\ Q(t) = Id. \end{cases}$$

Now we see that if $R(t)$ is given, $Q(t)$ can be determined. Then $l(t), \omega(t)$ are determined,

$$l(t) = Q(t)L(t), \quad A(\omega(t)) = Q(t)A(R(t))Q^T(t). \quad (3.9)$$

Then $\Lambda, X, Y, g_{ij}, g^{ij}, \Gamma, Mv, Nv$ can be determined as in section 2.

Moreover, if $L(t), R(t) \in C[0, T]$, then $l(t), \omega(t) \in C[0, T]$, with the estimate

$$|l(t)| + |\omega(t)| \leq C(T) (\|L\|_{L^\infty(0, T)} + \|R\|_{L^\infty(0, T)}). \quad (3.10)$$

Suppose that $(l^1(t), \omega^1(t))$ and $(l^2(t), \omega^2(t))$ are determined by $(L^1(t), R^1(t))$ and $(L^2(t), R^2(t))$ respectively in the above way, then

$$\begin{aligned}
&\|l_1 - l_2\|_{L^\infty(0, T)} + \|\omega_1 - \omega_2\|_{L^\infty(0, T)} \\
&\leq C(T)(1 + \|R_1(t)\|_{L^\infty(0, T)} + \|R_2(t)\|_{L^\infty(0, T)}) \cdot (\|L_1 - L_2\|_{L^\infty(0, T)} + \|R_1 - R_2\|_{L^\infty(0, T)}).
\end{aligned} \quad (3.11)$$

Lemma 3.4. Assume that v is a function in $L^\infty(0, T; \tilde{X}_*)$ and s is a nonnegative integer. Suppose there exists $M_* > 0$, such that $\|v\|_{L^\infty(0, T; \tilde{X})} \leq M_*$. Let

$$L(t) = l_{v(t)}, \quad R(t) = \omega_{v(t)}.$$

Suppose $l(t), \omega(t)$ is given by $L(t), R(t)$ as in (3.9) and $\Lambda, X, Y, g_{ij}, g^{ij}, \Gamma$ are defined as in section 2. Then for every $t \in [0, T]$, the following estimates hold:

$$\|J_X(\cdot, t)\|_{W^{s, \infty}(\mathbb{R}^3)} \leq C(s, T, M_*), \quad \|J_Y(X(\cdot, t), t)\|_{W^{s, \infty}(\mathbb{R}^3)} \leq C(s, T, M_*), \quad (3.12)$$

$$\|\Lambda(X(\cdot, t), t)\|_{W^{s, \infty}(\mathbb{R}^3)} \leq C(s, T, M_*), \quad \|g_{ij}(\cdot, t)\|_{W^{s, \infty}(\mathbb{R}^3)} \leq C(s, T, M_*), \quad (3.13)$$

$$\|g^{ij}(\cdot, t)\|_{W^{s, \infty}(\mathbb{R}^3)} \leq C(s, T, M_*), \quad \|G^{-1}(\cdot, t)\|_{W^{s, \infty}(\mathbb{R}^3)} \leq C(s, T, M_*), \quad (3.14)$$

$$\|\Gamma(\cdot, t)\|_{W^{s, \infty}(\mathbb{R}^3)} \leq C(s, T, M_*), \quad (3.15)$$

where G^{-1} is the inverse of G .

Proof. For every $j = 1, 2$ or 3 , let $z(y, t) = \frac{\partial X}{\partial y_j}$,

$$\begin{cases} \frac{\partial z(y, t)}{\partial t} = \frac{\partial \Lambda}{\partial x}(X(y, t), t) \cdot z(y, t), \\ z(y, 0) = \vec{e}_j, \end{cases}$$

where \vec{e}_j is the j -th vector of the basis of \mathbb{R}^3 . Then

$$z(y, t) = \vec{e}_j + \int_0^t \frac{\partial \Lambda}{\partial x}(X(y, s), s) \cdot z(y, s) ds. \quad (3.16)$$

It follows from Gronwall's inequality that $|z(y, t)| \leq C(T, M_*)$.

Since $\det(J_X) = 1$, then $J_Y = (J_X^{ij})$, where J_X^{ij} is the cofactor of J_X . Hence

$$|J_Y(X(\cdot, t), t)| \leq C(T, M_*).$$

Furthermore,

$$|G(\cdot, t)| \leq C(T, M_*), \quad |G^{-1}(\cdot, t)| \leq C(T, M_*).$$

Denote $D^\beta = \frac{\partial^\beta}{\partial y^\beta}$. From (3.16), one can deduce that

$$D^\alpha z = \int_0^t \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \left(\frac{\partial \Lambda}{\partial x} \right) D^{\alpha-\beta} z ds, \quad |\alpha| \leq s.$$

Following the preceding process, one can get the estimates (3.12)-(3.15). \square

Next lemma is about the Lipschitz continuity of the coefficients with respect to v .

Lemma 3.5. *Assume that the assumptions of Lemma 3.4 hold for v^i , $i = 1, 2$. Let $L^i(t) = l_{v^i(t)}$ and $R^i(t) = \omega_{v^i(t)}$. Define $Q^i(t), l^i(t), \omega^i(t)$ and other terms correspondingly. Let $L(t) = L^1(t) - L^2(t)$, $R(t) = R^1(t) - R^2(t)$, $X = X^1 - X^2$, $\Lambda(y, t) = \Lambda(X^1(y, t), t) - \Lambda(X^2(y, t), t)$, $G = (g^{ij}) = (g^{ij,1} - g^{ij,2})$, $g_{ij} = g_{ij}^1 - g_{ij}^2$, $G^{-1} = (G^1)^{-1} - (G^2)^{-1}$, and $\Gamma_{m,k}^j = \Gamma_{m,k}^{j,1} - \Gamma_{m,k}^{j,2}$. Then for every $t \in [0, T]$,*

$$\|X(\cdot, t)\|_{W^{s,\infty}(\mathbb{R}^3)} \leq C(s, T, M_*) (\|L\|_{L^\infty(0,T)} + \|R\|_{L^\infty(0,T)}), \quad (3.17)$$

$$\|\Lambda(\cdot, t)\|_{W^{s,\infty}(\mathbb{R}^3)} \leq C(s, T, M_*) (\|L\|_{L^\infty(0,T)} + \|R\|_{L^\infty(0,T)}), \quad (3.18)$$

$$\|g_{ij}(\cdot, t)\|_{W^{s,\infty}(\mathbb{R}^3)} \leq C(s, T, M_*) (\|L\|_{L^\infty(0,T)} + \|R\|_{L^\infty(0,T)}), \quad (3.19)$$

$$\|g^{ij}(\cdot, t)\|_{W^{s,\infty}(\mathbb{R}^3)} \leq C(s, T, M_*) (\|L\|_{L^\infty(0,T)} + \|R\|_{L^\infty(0,T)}), \quad (3.20)$$

$$\|G^{-1}(\cdot, t)\|_{W^{s,\infty}(\mathbb{R}^3)} \leq C(s, T, M_*) (\|L\|_{L^\infty(0,T)} + \|R\|_{L^\infty(0,T)}), \quad (3.21)$$

$$\|\Gamma(\cdot, t)\|_{W^{s,\infty}(\mathbb{R}^3)} \leq C(s, T, M_*) (\|L\|_{L^\infty(0,T)} + \|R\|_{L^\infty(0,T)}). \quad (3.22)$$

Proof. Since

$$\begin{cases} \frac{\partial X(y,t)}{\partial t} = \Lambda(X^1(y,t), t) - \Lambda(X^2(y,t), t), \\ X(y, 0) = 0, \end{cases}$$

Simple calculation and the estimates in Lemma 3.4 induce that

$$\begin{aligned} |\Lambda(X^1(y,t), t) - \Lambda(X^2(y,t), t)| &\leq C(T, M_*) |X^1(y,t) - X^2(y,t)| \\ &\quad + C(T, M_*) (|L(t)| + |R(t)|). \end{aligned}$$

Therefore,

$$\|X\|_{C([0,T]; L^\infty(\mathbb{R}^3))} \leq C(T, M_*) (\|L\|_{L^\infty(0,T)} + \|R\|_{L^\infty(0,T)}).$$

$$\begin{aligned} \|\Lambda(y, t)\|_{L^\infty(\mathbb{R}^3)} &\leq C(T, M_*) \|X^1(y, t) - X^2(y, t)\|_{L^\infty(\mathbb{R}^3)} \\ &\leq C(T, M_*) (\|L\|_{L^\infty(0,T)} + \|R\|_{L^\infty(0,T)}). \end{aligned}$$

Other estimates can be derived similarly. \square

4 A Priori H^s -estimates of ∇q

In the following text, $s > \frac{5}{2}$. Given a function $v \in C_w(0, T; H_s \cap \tilde{X}_*)$, which satisfies that $\|v\|_{L^\infty(0,T; H_s)} \leq M_0$. Suppose v is a solution to (2.18)-(2.25), taking the divergence of Eq.

(2.18), then q satisfies the following system,

$$\begin{cases} \operatorname{div} \left(\sum_{j=1}^3 g^{ij} \frac{\partial q}{\partial y_j} \right) = -\operatorname{div}(Mv + Nv), & \text{in } \Omega, \\ \sum_{i,j=1}^3 g^{ij} \frac{\partial q}{\partial y_j} n_i + \left(\frac{1}{m} \int_{\partial\Omega} q \vec{n} d\sigma \right) \cdot \vec{n} + \left(\bar{J}^{-1} \int_{\partial\Omega} y \times q \vec{n} d\sigma \right) \times y \cdot \vec{n} \\ = -(Mv + Nv) \cdot \vec{n} + (\omega_v \times l_v) \cdot \vec{n} - [\bar{J}^{-1}(\bar{J}\omega_v \times \omega_v)] \times y \cdot \vec{n}, & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

Here Q, l, ω, g^{ij}, Mv and Nv are given as in (3.9), (2.15)-(2.17) and (2.25), replacing L, R by l_v, ω_v .

In this section, we will give the H^s -estimates of ∇q , for every fixed time $t \in [0, T]$. For simplicity of writing, we omit t . Here is the main result of this section.

Proposition 4.1.

$$\|\nabla q\|_{H^s(\Omega)} \leq C(T, M_0)(1 + \|v\|_{H^s}),$$

with some constant C depending on T and M_0 .

Remark 4.2. *The system q satisfies is almost a classical elliptic problem with Neumann boundary condition. During the proof of Proposition 4.1, the main idea is to use the Lax-Milgram theorem to get the existence of weak solution, and the standard high-order regularity estimates for exterior elliptic problems.*

Proof. For every fixed $t \in [0, T]$, the matrix $G = (g^{ij}) = J_Y J_Y^T$, so G is positive definite. Denote $\lambda_i(y, t) > 0, (i = 1, 2, 3)$ the eigenvalues of the matrix (g^{ij}) . Since $\det(g^{ij}) = 1$, thus $\prod_{i=1}^3 \lambda_i = 1$ and $\sum_{i=1}^3 \lambda_i = \sum_{i=1}^3 g^{ii} > 0$. Let $\gamma_0 = \sup_{y \in \mathbb{R}^3} |g^{ii}|$, then we have $3\gamma_0 \geq \lambda_i \geq \frac{1}{(3\gamma_0)^2}$ for every $i = 1, 2, 3$. By virtue of Lemma 3.4, there exist constants $C_1(T, M_0)$ and $C_2(T, M_0)$,

$$C_1(T, M_0) \leq |\lambda_i| \leq C_2(T, M_0), \quad i = 1, 2, 3.$$

Next, we shall use the Lax-Milgram theorem to prove the existence of the solution of (4.1), and then give the H^s -estimate of this solution.

Set a bilinear form B and a linear functional F on $\dot{D}^{1,2}(\Omega)$ as follows, for every $\eta, q \in$

$\dot{D}^{1,2}(\Omega)$,

$$\begin{aligned}
B(q, \eta) &= \sum_{i=1}^3 \left(\sum_{j=1}^3 g^{ij} \frac{\partial q}{\partial y_j}, \frac{\partial \eta}{\partial y_i} \right)_{L^2(\Omega)} + \frac{1}{m} \left(\int_{\partial\Omega} q \vec{n} d\sigma \right) \cdot \left(\int_{\partial\Omega} \xi \vec{n} d\sigma \right) \\
&\quad + \left(\bar{J}^{-1} \int_{\partial\Omega} y \times q \vec{n} d\sigma \right) \cdot \left(\int_{\partial\Omega} y \times \eta \vec{n} d\sigma \right), \\
F(\eta) &= - \int_{\Omega} (Mv + Nv) \cdot \nabla \eta dy + \int_{\partial\Omega} (\omega_v \times l_v) \cdot \eta \vec{n} d\sigma \\
&\quad - \int_{\partial\Omega} [\bar{J}^{-1}(\bar{J}\omega_v \times \omega_v)] \times y \cdot \eta \vec{n} d\sigma.
\end{aligned}$$

Note that

$$\begin{aligned}
B(q, q) &= \sum_{i=1}^3 \left(\sum_{j=1}^3 g^{ij} \frac{\partial q}{\partial y_j}, \frac{\partial q}{\partial y_i} \right)_{L^2(\Omega)} + \frac{1}{m} \left(\int_{\partial\Omega} q \vec{n} d\sigma \right)^2 + \bar{J}w(q) \cdot w(q) \\
&\geq C_1(T, M_0) \|\nabla q\|_{L^2(\Omega)}^2 + \frac{1}{m} \left(\int_{\partial\Omega} q \vec{n} d\sigma \right)^2 + \bar{J}w(q) \cdot w(q),
\end{aligned} \tag{4.2}$$

where $w(q) = \bar{J}^{-1} \int_{\partial\Omega} y \times q \vec{n} d\sigma$.

Since \bar{J} is a positive definite matrix, then there exists some constant $a > 0$ such that

$$a^{-1}|w(q)|^2 \leq \bar{J}w(q) \cdot w(q) \leq a|w(q)|^2.$$

Combining the above inequality and (4.2), one gets that B is coercive.

On the other hand,

$$\left| \sum_{i=1}^3 \left(\sum_{j=1}^3 g^{ij} \frac{\partial q}{\partial y_j}, \frac{\partial \eta}{\partial y_i} \right)_{L^2(\Omega)} \right| \leq \|G\|_{L^\infty(\mathbb{R}^3)} \|\nabla q\|_{L^2(\Omega)} \|\nabla \eta\|_{L^2(\Omega)}.$$

Then along the line of the proposition 3.3.1 in [25], one can easily verify that the bilinear form B is bounded.

Now we turn to the functional F .

$$\begin{aligned}
&\left| - \int_{\Omega} (Mv + Nv) \cdot \nabla \eta dx \right| \\
&\leq \|Mv + Nv\|_{L^2(\Omega)} \cdot \|\nabla \eta\|_{L^2(\Omega)} \\
&\leq C (\|\Lambda\|_{W^{1,\infty}(\Omega)} + \|J_Y\|_{L^\infty(\Omega)} + \|J_X\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \\
&\quad + \|\Gamma\|_{L^\infty(\Omega)}) \cdot \|v\|_{H^1(\Omega)} \cdot \|\nabla \eta\|_{L^2(\Omega)} \\
&\leq C(T, M_0) \|\nabla \eta\|_{L^2(\Omega)},
\end{aligned} \tag{4.3}$$

Choosing some $\eta \in D^{1,2}(\Omega)$ such that $\int_{\Omega_r} \eta dy = 0$, for some large r .

$$\begin{aligned}
& \left| \int_{\partial\Omega} \{ \omega_v \times l_v - [\bar{J}^{-1}(\bar{J}\omega_v \times \omega_v)] \times y \} \cdot \eta \vec{n} d\sigma \right| \\
& \leq C(|\omega_v||l_v| + |\omega_v|^2) \|\eta\|_{L^2(\partial\Omega)} \\
& \leq C(\Omega)(|\omega_v||l_v| + |\omega_v|^2) \|\eta\|_{H^1(\Omega_r)} \\
& \leq C(T, M_0, \Omega) \|\nabla \eta\|_{L^2(\Omega)},
\end{aligned} \tag{4.4}$$

where the last inequality is given by the Poincaré's inequality.

From the above estimates, it follows that F is bounded. According to Lax-Milgram Theorem, there exists a unique $q \in \dot{D}^{1,2}(\Omega)$ such that

$$B(q, \eta) = F(\eta), \quad \forall \eta \in \dot{D}^{1,2}(\Omega)$$

with

$$\|\nabla q\|_{L^2(\Omega)} \leq C(T, M_0). \tag{4.5}$$

Let

$$L_1 = \frac{1}{m} \int_{\partial\Omega} q \vec{n} d\sigma, \quad w = \bar{J}^{-1} \int_{\partial\Omega} y \times q \vec{n} d\sigma.$$

Then

$$|L_1| \leq C \|\nabla q\|_{L^2(\Omega)} \leq C(T, M_0), \quad |w| \leq C \|\nabla q\|_{L^2(\Omega)} \leq C(T, M_0). \tag{4.6}$$

Now we go to the H^s -estimate of ∇q . Similar to [18], the method is a standard regularity estimate for an exterior problem of elliptic equations. Consider the Neumann system which is equivalent to (4.1),

$$\begin{cases} \displaystyle \operatorname{div} \left(\sum_{j=1}^3 g^{ij} \frac{\partial q}{\partial y_j} \right) = -\operatorname{div}(Mv + Nv), & \text{in } \Omega, \\ \displaystyle \sum_{i,j=1}^3 g^{ij} \frac{\partial q}{\partial y_j} n_i = -(Mv + Nv) \cdot \vec{n} - (\omega_v \times l_v) \cdot \vec{n} \\ \quad + [\bar{J}^{-1}(\bar{J}\omega_v \times \omega_v)] \times y \cdot \vec{n} - L_1 \cdot \vec{n} - (w \times y) \cdot \vec{n}, & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

To estimate $\|\nabla q\|_{H^s(\Omega)}$, the key is to estimate the terms $\|\operatorname{div}(Mv + Nv)\|_{H^{s-1}(\Omega)}$ and $\|-(Mv + Nv) \cdot \vec{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)}$.

$$\begin{aligned}
& \left\| \operatorname{div} \left(\left(\frac{\partial Y}{\partial t} + v \right) \cdot \nabla v \right) \right\|_{H^{s-1}(\Omega)} \\
&= \left\| \sum_{i,j=1}^3 \frac{\partial}{\partial y_i} \left(\frac{\partial Y_j}{\partial t} + v_j \right) \frac{\partial v_i}{\partial y_j} \right\|_{H^{s-1}(\Omega)} \\
&\leq \left\| \sum_{i,j,k=1}^3 \frac{\partial^2 Y_j}{\partial t \partial x_k} \frac{\partial X_k}{\partial y_i} \frac{\partial v_i}{\partial y_j} \right\|_{H^{s-1}(\Omega)} + \left\| \sum_{i,j=1}^3 \frac{\partial v_j}{\partial y_i} \frac{\partial v_i}{\partial y_j} \right\|_{H^{s-1}(\Omega)} \\
&\leq C \left(\|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)} \|J_X\|_{W^{s-1,\infty}(\mathbb{R}^3)} \|\nabla v\|_{H^{s-1}(\Omega)} + \|\nabla v\|_{H^{s-1}(\Omega)} \|v\|_{H^s(\Omega)} \right),
\end{aligned}$$

and

$$\begin{aligned}
& \left\| \sum_{i=1}^3 \frac{\partial}{\partial y_i} \left(\sum_{j,k=1}^3 \left\{ \Gamma_{j,k}^i \frac{\partial Y_k}{\partial t} + \frac{\partial Y_k}{\partial x_k} \frac{\partial^2 X_k}{\partial t \partial y} \right\} v_j + \sum_{j,k=1}^3 \Gamma_{j,k}^i v_j v_k \right) \right\|_{H^{s-1}(\Omega)} \\
&\leq C \left[(\|\Gamma\|_{W^{s,\infty}(\mathbb{R}^3)} \|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)} + \|J_Y\|_{W^{s,\infty}(\mathbb{R}^3)} \|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)}) \cdot \|v\|_{H^s(\Omega)} \right. \\
&\quad \left. + \|\Gamma\|_{W^{s,\infty}(\mathbb{R}^3)} \|v\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)} \right].
\end{aligned}$$

Hence,

$$\|\operatorname{div}(Mv + Nv)\|_{H^{s-1}(\Omega)} \leq C(T, M_0) \|v\|_{H^s}. \quad (4.8)$$

Denote

$$I = \left(\sum_{j,k=1}^3 \left\{ \Gamma_{j,k}^i \frac{\partial Y_k}{\partial t} + \frac{\partial Y_k}{\partial x_k} \frac{\partial^2 X_k}{\partial t \partial y} \right\} v_j + \sum_{j,k=1}^3 \Gamma_{j,k}^i v_j v_k \right)_i.$$

Then

$$-(Mv + Nv) \cdot \vec{n} = \left(\frac{\partial Y}{\partial t} + v \right) \cdot \nabla v \cdot \vec{n} + I \cdot \vec{n}.$$

Note that there is no derivative of v in the term I , so it is easy to handle.

$$\begin{aligned}
\|I \cdot \vec{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)} &\leq C \|I\|_{H^s(\Omega)} \\
&\leq C \left[(\|\Gamma\|_{W^{s,\infty}(\mathbb{R}^3)} \|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)} + \|J_Y\|_{W^{s,\infty}(\mathbb{R}^3)} \|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)}) \right. \\
&\quad \left. \cdot \|v\|_{H^s(\Omega)} + \|\Gamma\|_{W^{s,\infty}(\mathbb{R}^3)} \|v\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)} \right] \\
&\leq C(T, M_0) \|v\|_{H^s}.
\end{aligned} \quad (4.9)$$

To estimate $\left\| \left(\frac{\partial Y}{\partial t} + v \right) \cdot \nabla v \cdot \vec{n} \right\|_{H^{s-\frac{1}{2}}(\partial\Omega)}$, we proceed as in [1, 18]. Note $\left(\frac{\partial Y}{\partial t} + v \right) \cdot \vec{n} = 0$

on $\partial\Omega$, then

$$\begin{aligned}
& \left(\frac{\partial Y}{\partial t} + v \right) \cdot \nabla v \cdot \vec{n} \\
&= \left(\frac{\partial Y}{\partial t} + v \right) \cdot \nabla \left(\frac{\partial Y}{\partial t} + v \right) \cdot \vec{n} - \left(\frac{\partial Y}{\partial t} + v \right) \cdot \nabla \left(\frac{\partial Y}{\partial t} \right) \cdot \vec{n} \\
&= - \sum_{i,j=1}^3 \left(\frac{\partial Y}{\partial t} \right)_i \left(\frac{\partial Y}{\partial t} + v \right)_j \partial_i n_j - \left(\frac{\partial Y}{\partial t} + v \right) \cdot \nabla \left(\frac{\partial Y}{\partial t} \right) \cdot \vec{n}.
\end{aligned}$$

Hence,

$$\left\| \left(\frac{\partial Y}{\partial t} + v \right) \cdot \nabla v \cdot \vec{n} \right\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C \left(\|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)}^2 + 1 \right) (1 + \|v\|_{H^s(\Omega)}). \quad (4.10)$$

Combining (4.9) and (4.10), one has

$$\| - (Mv + Nv) \cdot \vec{n} \|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C(T, M_0)(1 + \|v\|_{H^s}). \quad (4.11)$$

The other terms can be estimated as follows:

$$\| [\bar{J}^{-1}(\bar{J}\omega \times \omega)] \times y \cdot \vec{n} \|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C(\Omega, R)|\omega|^2 \leq C(T, M_0), \quad (4.12)$$

$$\| L_1 \cdot \vec{n} \|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C|L_1| \leq C(T, M_0), \quad (4.13)$$

$$\| (\omega_v \times l_v) \cdot \vec{n} \|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C(\Omega, R)|\omega||l| \leq C(T, M_0), \quad (4.14)$$

$$\| (w \times y) \cdot \vec{n} \|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C(\Omega, R)|w| \leq C(T, M_0). \quad (4.15)$$

Choose some $r > 0$ such that $\text{supp}(\xi) \subset B_{\frac{r}{2}}$, and a cut-off function ξ_1 ,

$$\xi_1(y) = \begin{cases} 1, & \text{if } |y| \leq 2r, \\ 0, & \text{if } |y| \geq 3r. \end{cases}$$

Hence, $p_1 = \xi_1 q$ solves the following equation

$$\left\{ \begin{array}{ll} \operatorname{div} \left(\sum_{j=1}^3 g^{ij} \frac{\partial p_1}{\partial y_j} \right) = -\xi_1 \operatorname{div}(Mv + Nv) + \sum_{i,j=1}^3 \frac{\partial g^{ij}}{\partial y_i} \frac{\partial \xi_1}{\partial y_j} q \\ \quad + \sum_{i,j=1}^3 g^{ij} \left(\frac{\partial \xi_1}{\partial y_j} \frac{\partial q}{\partial y_i} + \frac{\partial \xi_1}{\partial y_i} \frac{\partial q}{\partial y_j} + \frac{\partial^2 \xi_1}{\partial y_i \partial y_j} q \right), & \text{in } B_{4r} \setminus \mathcal{O}, \\ \sum_{i,j=1}^3 g^{ij} \frac{\partial p_1}{\partial y_j} n_i = -(Mv + Nv) \cdot \vec{n} - \omega_v \times l_v \cdot \vec{n} \\ \quad + [\bar{J}^{-1}(\bar{J}\omega_v \times \omega_v)] \times y \cdot \vec{n} - L_1 \cdot \vec{n} - w \times y \cdot \vec{n}, & \text{on } \partial\mathcal{O}, \\ \sum_{i,j=1}^3 g^{ij} \frac{\partial p_1}{\partial y_j} n_i = 0, & \text{on } \partial B_{4r}. \end{array} \right. \quad (4.16)$$

By virtue of the regularity theory for elliptic equations[22], for any $\beta \geq 1$,

$$\begin{aligned} \|p_1\|_{H^{\beta+1}(B_{4r} \setminus \mathcal{O})} &\leq h_1 \left(\|G\|_{W^{s,\infty}(\mathbb{R}^3)}, (3\gamma_0)^2 \right) \left(\left\| \sum_{i,j=1}^3 \frac{\partial g^{ij}}{\partial y_i} \frac{\partial \xi_1}{\partial y_j} q \right\|_{H^{\beta-1}(B_{4r} \setminus \mathcal{O})} \right. \\ &\quad + \|\xi_1 \operatorname{div}(Mv + Nv)\|_{H^{\beta-1}(B_{4r} \setminus \mathcal{O})} + \left\| \sum_{i,j=1}^3 g^{ij} \left(\frac{\partial \xi_1}{\partial y_j} \frac{\partial q}{\partial y_i} + \frac{\partial \xi_1}{\partial y_i} \frac{\partial q}{\partial y_j} \right) \right\|_{H^{\beta-1}(B_{4r} \setminus \mathcal{O})} \\ &\quad + \left\| \sum_{i,j=1}^3 g^{ij} \frac{\partial^2 \xi}{\partial y_i \partial y_j} q \right\|_{H^{\beta-1}(B_{4r} \setminus \mathcal{O})} + \left\| -(Mv + Nv) \cdot \vec{n} \right\|_{H^{\beta-\frac{1}{2}}(\partial\Omega)} \\ &\quad \left. + \left\| \bar{J}^{-1}(\bar{J}\omega \times \omega) \cdot \vec{n} \right\|_{H^{\beta-\frac{1}{2}}(\partial\Omega)} + \left\| (L_1 + w \times y) \cdot \vec{n} \right\|_{H^{\beta-\frac{1}{2}}(\partial\Omega)} + \|p_1\|_{L^2(B_{4r} \setminus \mathcal{O})} \right), \end{aligned} \quad (4.17)$$

where $h_1(\cdot, \cdot)$ can be chosen an increasing function with respect to both variables. In fact,

$$\begin{aligned} &\left\| \sum_{i,j=1}^3 \frac{\partial g^{ij}}{\partial y_i} \frac{\partial \xi_1}{\partial y_j} q \right\|_{H^{\beta-1}(B_{4r} \setminus \mathcal{O})} + \left\| \sum_{i,j=1}^3 g^{ij} \left(\frac{\partial \xi_1}{\partial y_j} \frac{\partial q}{\partial y_i} + \frac{\partial \xi_1}{\partial y_i} \frac{\partial q}{\partial y_j} \right) \right\|_{H^{\beta-1}(B_{4r} \setminus \mathcal{O})} \\ &\quad + \left\| \sum_{i,j=1}^3 g^{ij} \frac{\partial^2 \xi}{\partial y_i \partial y_j} q \right\|_{H^{\beta-1}(B_{4r} \setminus \mathcal{O})} \\ &\leq C \left(\|G\|_{W^{s,\infty}(\mathbb{R}^3)} \cdot \|q\|_{H^{\beta-1}(B_{3r} \setminus B_{2r})} + \|G\|_{W^{\beta-1,\infty}(\mathbb{R}^3)} \cdot \|\nabla q\|_{H^{\beta-1}(B_{3r} \setminus B_{2r})} \right). \end{aligned} \quad (4.18)$$

Combining the above estimates, one gets

$$\begin{aligned} &\|q\|_{H^{\beta+1}(B_{2r} \setminus \mathcal{O})} \\ &\leq C(T, M_0, r) \left(\|q\|_{H^{\beta-1}(B_{3r} \setminus B_{2r})} + \|q\|_{L^2(B_{3r} \setminus \mathcal{O})} + 1 \right). \end{aligned} \quad (4.19)$$

Choose some particular q such that

$$\int_{B_{4r} \setminus \mathcal{O}} q(y) dy = 0.$$

It is reasonable, since q is still a solution to (4.7) if it is added by a constant. By Poincaré's inequality,

$$\begin{aligned} \|q\|_{H^{\beta+1}(B_{2r})} &\leq C(T, M_0, r) \left(\|q\|_{H^\beta(B_{3r} \setminus B_{2r})} + \|\nabla q\|_{L^2(B_{4r} \setminus \mathcal{O})} + 1 \right) \\ &\leq C(T, M_0, r) \left(\|q\|_{H^\beta(B_{3r} \setminus B_{2r})} + 1 \right). \end{aligned} \quad (4.20)$$

It implies that high-order regularity of q can be controlled by the lower-order regularity. Therefore, using this method by choosing appropriate r , we can get that for every $R > 0$, such that $\mathcal{O} \subseteq B_{\frac{R}{2}}$,

$$\|\nabla q\|_{H^s(\Omega_R)} \leq C(T, M_0, R)(1 + \|v\|_{H_s}). \quad (4.21)$$

Fix some R big enough. Choose some smooth cut-off function ξ_2 , such that

$$\xi_2(y) = \begin{cases} 0, & \text{if } |y| \leq \frac{3}{4}R, \\ 1, & \text{if } |y| \geq R. \end{cases}$$

Since $g^{ij} = \delta_{ij}$ outside $B_{\frac{R}{2}}$, hence

$$\operatorname{div}(G \cdot \nabla q) = \Delta q.$$

Let $p_2 = \xi_2 q$, then

$$\Delta p_2 = \xi_2(-\operatorname{div}(Mv + Nv)) + 2\nabla \xi_2 \cdot \nabla q + \Delta \xi_2 q := \tilde{f}. \quad (4.22)$$

Therefore,

$$\|\nabla p_2\|_{H^s(\mathbb{R}^3)} \leq C \left(\|\tilde{f}\|_{H^{s-1}(\mathbb{R}^3)} + \|\nabla p_2\|_{L^2(\mathbb{R}^3)} \right). \quad (4.23)$$

\tilde{f} is estimated as follows,

$$\begin{aligned} &\|\tilde{f}\|_{H^{s-1}(\mathbb{R}^3)} \\ &\leq C \left(\|\operatorname{div}(Mv + Nv)\|_{H^{s-1}(\Omega)} + \|\nabla q\|_{H^{s-1}(\frac{R}{2} \leq |y| \leq R)} + \|q\|_{H^{s-1}(\frac{R}{2} \leq |y| \leq R)} \right) \\ &\leq C(T, M_0)(1 + \|v\|_{H_s}). \end{aligned} \quad (4.24)$$

Hence

$$\|\nabla q\|_{H^s(\mathbb{R}^3 \setminus B_R)} \leq C(T, M_0)(1 + \|v\|_{H_s}). \quad (4.25)$$

(4.21) and (4.25) give that

$$\|\nabla q\|_{H^s(\Omega)} \leq C(T, M_0)(1 + \|v\|_{H_s}). \quad (4.26)$$

It completes the proof of Proposition 4.1. \square

5 Construction of approximate solutions

In this section, we will construct a sequence of approximate solutions. Similar to [18], the main idea is the Kato-Lai theory. Suppose for every $n \geq 0$, there is a function $v^n(t) \in C_w([0, T]; H_s)$. Denote $L^n(t) = l_{\mathbb{P}v^n(t)}$, $R^n(t) = \omega_{\mathbb{P}v^n(t)}$. Then $L^n(t), R^n(t) \in C[0, T]$. Solving the following initial value problem

$$\begin{cases} \frac{dQ^n(t)}{dt} = Q^n(t)A(R^n(t)), \\ Q^n(0) = Id. \end{cases} \quad (5.1)$$

One can get a solution $Q^n(t)$.

Define

$$\begin{aligned} l^n(t) &= Q^n(t)L^n(t), \quad A(\omega^n(t)) = Q^n(t)A(R^n(t))[Q^n(t)]^T, \\ \psi^n &= \xi((Q^n)^T(t)(x - h^n(t))), \quad h^n = \int_0^t Q^n(s)l^n(s)ds, \end{aligned}$$

and

$$\begin{aligned} V^n &= l^n(t) + \omega^n(t) \times (x - h^n(t)), \\ W^n &= l^n(t) \times (x - h^n(t)) + \frac{|x - h^n(t)|^2}{2} \omega^n(t), \end{aligned}$$

where ξ is a cut-off function given in section 2.

Let $\Lambda^n = \psi^n V^n + \nabla \psi^n W^n$. Hence one can define $X^n(\cdot, t)$, $\Lambda^n(X^n(\cdot, t), t)$, $g^{ij,n}$, g_{ij}^n , $\Gamma_{jk}^{i,n}$, M^n, N^n given in (2.17) and (2.25). Suppose that q^n is the solution to the following system,

$$\begin{cases} \operatorname{div} \left(\sum_{j=1}^3 g^{ij,n} \frac{\partial q^n}{\partial y_j} \right) = -\operatorname{div}(M^n \mathbb{P}v^n + N^n \mathbb{P}v^n), & \text{in } \Omega \\ \sum_{i,j=1}^3 g^{ij,n} \frac{\partial q^n}{\partial y_j} n_i + \frac{1}{m} \left(\int_{\partial\Omega} q^n \vec{n} d\sigma \right) \cdot \vec{n} + \left[\bar{J}^{-1} \int_{\partial\Omega} y \times q^n \vec{n} d\sigma \right] \times y \cdot \vec{n} \\ = -(M^n \mathbb{P}v^n + N^n \mathbb{P}v^n) \cdot \vec{n} - (R^n \times L^n) \cdot \vec{n} + [\bar{J}^{-1}(\bar{J}R^n \times R^n)] \times y \cdot \vec{n}, & \text{on } \partial\Omega. \end{cases} \quad (5.2)$$

Now define an operator $A^n(t, v)$ as in [18],

$$\begin{aligned}
A^n(t, v) = & \vec{1}_\Omega \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla v - \mathcal{Q} \left[\vec{1}_\Omega \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla \mathbb{P}v \right] \\
& + \mathbb{P} \left[\vec{1}_\Omega \left(\sum_{j,k=1}^3 \left\{ \Gamma_{j,k}^{i,n-1} \frac{\partial Y_k^{n-1}}{\partial t} + \frac{\partial Y^{n-1}}{\partial x_k} \frac{\partial^2 X_k^{n-1}}{\partial t \partial y_j} \right\} (\mathbb{P}v)_j^{n-1} + \sum_{j,k=1} \Gamma_{j,k}^{i,n-1} (\mathbb{P}v)_j^{n-1} (\mathbb{P}v)_k^{n-1} \right) \right] \\
& \mathbb{P} \left[\vec{1}_\Omega \left(\sum_{j=1}^3 g^{ij,n-1} \frac{\partial q^{n-1}}{\partial y_j} \right) \right] + \mathbb{P} \left[\vec{1}_\mathcal{O} (R^{n-1} \times L^{n-1} - \bar{J}^{-1}(\bar{J}R^{n-1} \times R^{n-1}) \times y) \right] \\
& + \mathbb{P} \left[\vec{1}_\mathcal{O} \left(-\frac{1}{m} \int_{\partial\Omega} q^{n-1} \vec{n} d\sigma - \left(\bar{J}^{-1} \int_{\partial\Omega} y \times q^{n-1} \vec{n} d\sigma \right) \times y \right) \right],
\end{aligned} \tag{5.3}$$

where the operator $\mathcal{Q} = I - \mathbb{P}$.

Let

$$v_0(y) = \begin{cases} u_0(y), & y \in \Omega, \\ l_0 + \omega_0 \times y, & y \in \mathcal{O}. \end{cases}$$

Consider the following Cauchy problem,

$$\begin{cases} v_t^n + A^n(t, v^n) = 0, \\ v^n(0) = v_0, \end{cases} \tag{5.4}$$

where $v_0 \in H_s \cap \tilde{X}_*$.

In particular, let $v^0(y, t) = v_0(y)$. We shall prove that for each $n \in \mathbb{N}$, there exists a solution $v^n \in C_w(0, T_n; H_s)$ with some uniform lifespan T_n .

For simplicity, denote

$$\begin{aligned}
(F^{n-1}v)_i = & \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla v_i, \quad (G^{n-1} \cdot \nabla q^{n-1})_i = \sum_{j=1}^3 g^{ij,n-1} \frac{\partial q^{n-1}}{\partial y_j}, \\
(E^{n-1})_i = & \vec{1}_\Omega \left[\left(\sum_{j,k=1}^3 \left\{ \Gamma_{j,k}^{i,n-1} \frac{\partial Y_k^{n-1}}{\partial t} + \frac{\partial Y^{n-1}}{\partial x_k} \frac{\partial^2 X_k^{n-1}}{\partial t \partial y_j} \right\} (\mathbb{P}v^{n-1})_j \right. \right. \\
& \left. \left. + \sum_{j,k=1} \Gamma_{j,k}^{i,n-1} (\mathbb{P}v^{n-1})_j (\mathbb{P}v^{n-1})_k \right) \right], \\
(K^{n-1})_i = & \left[\vec{1}_\mathcal{O} \left(\frac{1}{m} R^{n-1} \times L^{n-1} - \bar{J}^{-1}(\bar{J}R^{n-1} \times R^{n-1}) \times y \right) \right]_i \\
& - \left\{ \vec{1}_\mathcal{O} \left[\frac{1}{m} \int_{\partial\Omega} q^{n-1} \vec{n} d\sigma - \left(\bar{J}^{-1} \int_{\partial\Omega} y \times q^{n-1} \vec{n} d\sigma \right) \times y \right] \right\}_i,
\end{aligned}$$

and denote $M_0 = 2\|v_0\|_{H_s}$. Suppose that there exists some $T_0 > 0$ such that for all $k < n$,

$$\|v^k\|_{L^\infty(0, T_0; H_s)} \leq M_0.$$

For the estimate of $(v, A^n(t, v))_{H_s}$,

$$\begin{aligned} |(v, A^n(t, v))_{H_s}| &\leq |(\vec{1}_\Omega F^{n-1} v, v)_{H_s}| + |(\mathcal{Q}(\vec{1}_\Omega F^{n-1} \mathbb{P} v), v)_{H_s}| + |(\mathbb{P} E^{n-1} v, v)_{H_s}| \\ &\quad + |(\mathbb{P}(\vec{1}_\Omega G^{n-1} \nabla q^{n-1}), v)_H| + |(\mathbb{P} K^{n-1}, v)_{H_s}| \\ &:= J_1 + J_2 + J_3 + J_4 + J_5 \end{aligned}$$

Then we estimate them term by term. Starting from the easiest one,

$$\begin{aligned} J_5 &\leq \|\mathbb{P} K^{n-1}\|_{H_s} \|v\|_{H_s} \\ &\leq C \|K^{n-1}\|_{H_s} \|v\|_{H_s} \\ &\leq C \left(\left| \int_{\partial\Omega} q^{n-1} \vec{n} d\sigma \right| + \left| \int_{\partial\Omega} y \times q^{n-1} \vec{n} d\sigma \right| + |L^{n-1}| |R^{n-1}| + |R^{n-1}|^2 \right) \|v\|_{H_s} \quad (5.5) \\ &\leq C (|L^{n-1}| \cdot |R^{n-1}| + |R^{n-1}|^2 + \|\nabla q^{n-1}\|_{L^2(\Omega)}) \cdot \|v\|_{H_s} \\ &\leq C(T_0, M_0) \|v\|_{H_s}. \end{aligned}$$

By Lemma 3.3 and Lemma 3.4,

$$\begin{aligned} J_4 &\leq \|\mathbb{P}(\vec{1}_\Omega(G^{n-1} \nabla q^{n-1}))\|_{H_s} \cdot \|v\|_{H_s} \\ &\leq C \|G^{n-1} \nabla q^{n-1}\|_{H^s(\Omega)} \cdot \|v\|_{H_s} \quad (5.6) \\ &\leq C(T_0, M_0) \|v\|_{H_s}. \end{aligned}$$

J_3 is also easy to estimate since there is no derivative of v or v^{n-1} ,

$$J_3 \leq C(T_0, M_0) \|v\|_{H_s}. \quad (5.7)$$

Now the most difficult terms J_1 and J_2 are left, since there is derivative of v or $\mathbb{P} v$.

$$J_1 = \left| \sum_{|\alpha| \leq s} \sum_{\alpha_1 \leq \alpha} \sum_{i=1}^3 \int_{\Omega} \partial^{\alpha_1} \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P} v^{n-1} \right) \cdot \nabla \partial^{\alpha-\alpha_1} v_i \partial^{\alpha} v_i dy \right|. \quad (5.8)$$

When $\alpha_1 = (0, 0, 0)$, since

$$\operatorname{div} \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P} v^{n-1} \right) = 0 \quad \text{in } \Omega, \quad \text{and} \quad \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P} v^{n-1} \right) \cdot \vec{n} = 0 \quad \text{on } \partial\Omega,$$

then

$$\int_{\Omega} \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P} v^{n-1} \right) \cdot \nabla \partial^{\alpha} v_i \partial^{\alpha} v_i dy = 0.$$

Therefore, we assume that $|\alpha_1| \geq 1$. Let $\alpha_2 = \alpha - \alpha_1$.

$$\begin{aligned}
& \left\| \partial^{\alpha_1} \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla \partial^{\alpha_2} v_i \right\|_{L^2(\Omega)} \\
& \leq \left\| \partial^{\alpha_1} \frac{\partial Y^{n-1}}{\partial t} \cdot \nabla \partial^{\alpha_2} v_i \right\|_{L^2(\Omega)} + \left\| \partial^{\alpha_1} \mathbb{P}v^{n-1} \cdot \nabla \partial^{\alpha_2} v_i \right\|_{L^2(\Omega)} \\
& \leq C(T_0, M_0) \|v\|_{H_s} + \|\mathbb{P}v^{n-1}\|_{L^\infty(\Omega)} \|\nabla v\|_{H^{s-1}(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} \|\mathbb{P}v^{n-1}\|_{H^s(\Omega)} \\
& \leq C(T_0, M_0) \|v\|_{H_s}.
\end{aligned}$$

Hence,

$$J_1 \leq C(T_0, M_0) \|v\|_{H_s}^2. \quad (5.9)$$

For the term J_2 ,

$$J_2 \leq \|\mathcal{Q}[\vec{1}_\Omega F^{n-1} \mathbb{P}v]\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)} + C\|F^{n-1} \mathbb{P}v\|_{L^2(\Omega)} \|v\|_{L^2(\mathcal{O})}.$$

Herein,

$$\begin{aligned}
\|F^{n-1} \mathbb{P}v\|_{L^2(\Omega)} \|v\|_{L^2(\mathcal{O})} & \leq \left\| \frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right\|_{L^\infty(\Omega)} \|\mathbb{P}v\|_{H_s} \|v\|_{L^2(\mathcal{O})} \\
& \leq C(T_0, M_0) \|v\|_{H_s}^2.
\end{aligned}$$

To estimate $\|\mathcal{Q}[\vec{1}_\Omega F^{n-1} \mathbb{P}v]\|_{H^s(\Omega)}$, consider the following system,

$$\begin{cases} \Delta \phi = \operatorname{div} \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla \mathbb{P}v, & \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} = \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla \mathbb{P}v \cdot \vec{n}, & \text{on } \partial\Omega. \end{cases}$$

In fact, $\mathcal{Q}[\vec{1}_\Omega F^{n-1} \mathbb{P}v] = \nabla \phi$.

Note that in the domain Ω ,

$$\operatorname{div} \left[\left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla \mathbb{P}v \right] = \sum_{i,j=1}^3 \partial_j \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v \right)_i \partial_i (\mathbb{P}v)_j,$$

and on the boundary $\partial\Omega$,

$$\begin{aligned}
\left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla \mathbb{P}v \cdot \vec{n} & = \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla (\mathbb{P}v - l_{\mathbb{P}v} - \omega_{\mathbb{P}v} \times y) \cdot \vec{n} \\
& + \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla (l_{\mathbb{P}v} + \omega_{\mathbb{P}v} \times y) \cdot \vec{n}.
\end{aligned}$$

Hence, as estimating ∇q in section 4, one can get that

$$\|\nabla \phi\|_{H^s(\Omega)} \leq C(T_0, M_0) \|v\|_{H_s},$$

consequently,

$$J_2 \leq C(T_0, M_0) \|v\|_{H_s}^2.$$

Therefore, we have

$$|(v, A^n(t, v))_{H_s}| \leq C(T_0, M_0)(1 + \|v\|_{H_s}^2). \quad (5.10)$$

Now, fix some big T_0 . Let's consider the corresponding ordinary differential equation,

$$\gamma'(t) = C(T_0, M_0)(1 + \gamma(t)), \quad \gamma(0) = \|v_0\|_{H_s}^2,$$

with $C(T_0, M_0)$ the same constant as in (5.10). Assume that $T \leq T_0$ is a time such that for every $t \in [0, T]$,

$$\gamma(t) \leq 4\|v_0\|_{H_s}^2 = (M_0)^2.$$

Then by the Kato-Lai theory, the solution v^n to (5.4) can be derived at least on $[0, T]$, i.e., $v^n \in C_w([0, T]; H_s) \cap C_w^1([0, T]; \tilde{X})$, and

$$\|v^n(t)\|_{H_s}^2 \leq \gamma(t) \leq (M_0)^2, \quad t \in [0, T].$$

For $n = 1$, we choose $v^0(y, t) = v_0(y)$. Following the above process, one can construct a solution v^1 to the system (5.4). By iterating the same steps, a sequence of approximate solutions $\{v^n\}$ can be constructed.

6 The convergence of approximate solutions

In this section, we show that $\{v^n\}$ converges to a solution of the system (2.18)-(2.24).

According to the estimates in section 5,

$$\|v^n\|_{L^\infty(0, T; H_s)} \leq M_0, \quad (6.1)$$

$$\|\partial_t v^n\|_{L^\infty(0, T; \tilde{X})} \leq M_1, \quad (6.2)$$

$$\|\nabla q^n\|_{L^\infty(0, T; H^{s-1}(\Omega))} \leq M_2. \quad (6.3)$$

Since \mathbb{P} is a bounded operator on H_s , \tilde{X} , and it commutes with ∂_t , then

$$\|\mathbb{P}v^n\|_{L^\infty(0, T; H_s)} \leq M_3, \quad (6.4)$$

$$\|\partial_t \mathbb{P}v^n\|_{L^\infty(0, T; \tilde{X})} \leq M_4. \quad (6.5)$$

Hence from the [21], there exists some function $v \in C_w([0, T]; H_s)$ such that,

$$v^n \rightarrow v \text{ in } C_w(0, T; H_s), \quad (6.6)$$

$$\mathbb{P}v^n \rightarrow \mathbb{P}v \text{ in } C_w(0, T; H_s). \quad (6.7)$$

By the Aubin-Lions lemma, for every r_0 large enough,

$$v^n \rightarrow v \text{ in } C([0, T]; H^{s-1}(\Omega_{r_0}) \cap L^2(B_{r_0})), \quad (6.8)$$

$$\mathbb{P}v^n \rightarrow \mathbb{P}v \text{ in } C([0, T]; H^{s-1}(\Omega_{r_0}) \cap L^2(B_{r_0})). \quad (6.9)$$

Moreover, (6.9) implies that

$$L^n(t) \rightarrow L(t) = l_{\mathbb{P}v(t)} \text{ in } C[0, T], \quad (6.10)$$

$$R^n(t) \rightarrow R(t) = \omega_{\mathbb{P}v(t)} \text{ in } C[0, T]. \quad (6.11)$$

While (6.3), (6.8) and (6.9) tell that there exists some function q such that

$$\nabla q^n \rightarrow \nabla q \text{ in } C_w([0, T]; H^{s-1}(\Omega)), \quad (6.12)$$

$$\int_{\partial\Omega} q^n \vec{n} d\sigma \rightarrow \int_{\partial\Omega} q \vec{n} d\sigma \text{ in } C[0, T], \quad (6.13)$$

$$\int_{\partial\Omega} y \times q^n \vec{n} d\sigma \rightarrow \int_{\partial\Omega} y \times q \vec{n} d\sigma \text{ in } C[0, T]. \quad (6.14)$$

In fact, q is a solution to the system (4.1). It can be seen by taking the limit of (5.2).

From all the convergence results (6.1)-(6.14), it follows that

$$\begin{cases} v_t + A(t, v) = 0, \\ v(0) = v_0, \end{cases} \quad (6.15)$$

where

$$\begin{aligned} A(t, v) = & \vec{I}_\Omega \left(\frac{\partial Y}{\partial t} + \mathbb{P}v \right) \cdot \nabla v - \mathcal{Q} \left[\vec{I}_\Omega \left(\frac{\partial Y}{\partial t} + \mathbb{P}v \right) \cdot \nabla \mathbb{P}v \right] \\ & + \mathbb{P} \left[\vec{I}_\Omega \left(\sum_{j,k=1} \left\{ \Gamma_{j,k} \frac{\partial Y_k}{\partial t} + \frac{\partial Y}{\partial x_k} \frac{\partial^2 X_k}{\partial t \partial y_j} \right\} (\mathbb{P}v)_j + \sum_{j,k=1} \Gamma_{j,k} (\mathbb{P}v)_j (\mathbb{P}v)_k \right) \right] \\ & + \mathbb{P} \left[\vec{I}_\Omega \left(\sum_{j=1}^3 g^j \frac{\partial q}{\partial y_j} \right) \right] + \mathbb{P} \left[\vec{I}_\mathcal{O} (R \times L + \bar{J}^{-1}(\bar{J}R \times R) \times y) \right] \\ & - \mathbb{P} \left[\vec{I}_\mathcal{O} \left(\frac{1}{m} \int_{\partial\Omega} q \vec{n} d\sigma - \left(\bar{J}^{-1} \int_{\partial\Omega} y \times q \vec{n} d\sigma \right) \times y \right) \right], \end{aligned}$$

Next, we shall prove that v is a solution of the systems (2.18)-(2.24). The proof starts with the observation that $v(t) = \mathbb{P}v(t)$, for all $t \in [0, T]$. In fact, applying \mathcal{Q} to each term in (6.15) and taking the inner product with $\mathcal{Q}v(t)$ in \tilde{X} yields

$$\frac{d}{dt} \frac{1}{2} \|\mathcal{Q}v(t)\|_{\tilde{X}}^2 + (\mathcal{Q}v(t), \mathcal{Q}A(t, v))_{\tilde{X}} = 0. \quad (6.16)$$

Note that $\operatorname{div} \left(\frac{\partial Y}{\partial t} + \mathbb{P}v \right) = 0$ in Ω and $\left(\frac{\partial Y}{\partial t} + \mathbb{P}v \right) \cdot \vec{n} = 0$ on $\partial\Omega$, then

$$\begin{aligned} (\mathcal{Q}v(t), \mathcal{Q}A(t, v))_{\tilde{X}} &= \left(\mathcal{Q}v(t), \vec{1}_\Omega \left(\frac{\partial Y}{\partial t} + \mathbb{P}v \right) \cdot \nabla \mathcal{Q}v(t) \right)_{\tilde{X}} \\ &= \int_\Omega \mathcal{Q}v \cdot \left(\left(\frac{\partial Y}{\partial t} + \mathbb{P}v \right) \cdot \nabla \mathcal{Q}v(t) \right) dy \\ &= 0 \end{aligned} \quad (6.17)$$

Since $v_0 = \mathbb{P}v_0$, it tells that $\mathcal{Q}v_0 = 0$. Hence, for every $t \in [0, T]$, $\mathcal{Q}v(t) = 0$. Therefore, (6.15) can be written as

$$\begin{aligned} &\frac{\partial v}{\partial t} + \mathbb{P}[\vec{1}_\Omega(Mv + Nv + G \cdot \nabla q)] + \\ &\mathbb{P} \left[\vec{1}_\mathcal{O} \left(\frac{1}{m} R \times L - \bar{J}^{-1}(\bar{J}R \times R) \times y \right) \right. \\ &\quad \left. - \vec{1}_\mathcal{O} \left(\frac{1}{m} \int_{\partial\Omega} q \vec{n} d\sigma - (\bar{J}^{-1} \int_{\partial\Omega} y \times q \vec{n} d\sigma) \times y \right) \right] = 0 \end{aligned} \quad (6.18)$$

Taking the inner product in \tilde{X} with a test function $\phi \in \tilde{X}_*$, one has

$$\begin{aligned} &\int_\Omega (v' + Mv + Nv + G \cdot \nabla q) \cdot \phi dy \\ &+ mL' \cdot l_\phi - \int_{\partial\Omega} q \vec{n} d\sigma \cdot l_\phi + m(R \times L) \cdot l_\phi \\ &+ \bar{J}R' \cdot \omega_\phi - \bar{J} [\bar{J}^{-1}(\bar{J}R \times R)] \cdot \omega_\phi - \bar{J} \left[\bar{J}^{-1} \int_{\partial\Omega} y \times q \vec{n} d\sigma \right] \cdot \omega_\phi \\ &= 0. \end{aligned} \quad (6.19)$$

For every function $\phi \in C_0^\infty(\mathbb{R}^3)$, with $\operatorname{supp}(\phi) \subseteq \Omega$, and $\operatorname{div} \phi = 0$ in \mathbb{R}^3 , (6.19) yields

$$\int_\Omega (v' + Mv + Nv + G \cdot \nabla q) \cdot \phi dy = 0,$$

After the theory of Helmholtz-Weyl decomposition, there exists a function p such that $\nabla p \in L^\infty(0, T; H^{s-1}(\Omega))$ and

$$v' + Mv + Nv + G \cdot \nabla q + \nabla p = 0 \quad \text{in } \Omega \times [0, T]. \quad (6.20)$$

From the identification of q and (6.20), one knows that for every $t \in [0, T]$,

$$\begin{cases} \Delta p = 0, & \text{in } \Omega, \\ \frac{\partial p}{\partial \vec{n}} = 0, & \text{on } \partial\Omega. \end{cases}$$

The above system has only constant solutions, thus

$$v' + Mv + Nv + G \cdot \nabla q = 0 \quad \text{in } \Omega \times [0, T]. \quad (6.21)$$

Now taking some test function $\phi(x) \in \tilde{X}$ such that $\phi(y) = l_\phi$ in \mathcal{O} , then

$$mL' \cdot l_\phi - \left(\int_{\partial\Omega} q \vec{n} d\sigma \right) \cdot l_\phi + (mR \times L) \cdot l_\phi = 0.$$

Since l_ϕ is arbitrary, then

$$mL' = \int_{\partial\Omega} q \vec{n} d\sigma - R \times L. \quad (6.22)$$

Similarly, taking some test function $\phi(y) \in \tilde{X}$ such that $\phi(y) = \omega_\phi \times y$ in \mathcal{O} , then

$$\bar{J}R' \cdot \omega_\phi - \bar{J}(\bar{J}^{-1}(\bar{J}R \times R)) \cdot \omega_\phi - \bar{J} \left(\bar{J}^{-1} \int_{\partial\Omega} y \times q \vec{n} d\sigma \right) \cdot \omega_\phi = 0.$$

Thus

$$\bar{J}R' = \int_{\partial\Omega} y \times q \vec{n} d\sigma + \bar{J}R \times R. \quad (6.23)$$

Therefore, $(v, q, L(t), R(t))$ is a solution to the system (2.18)-(2.24).

7 Uniqueness and continuity with respect to time

In this section, we will prove that the solution of the system (2.18)-(2.24) is unique and then get the continuity in H_s with respect to time.

Assume that there exist two solutions $v^1, v^2 \in C_w([0, T]; H_s) \cap C_w^1([0, T]; \tilde{X})$ to the system (2.18)-(2.24), then

$$v_t^1 + M^1 v^1 + N^1 v^1 + G^1 \nabla q^1 = 0, \quad \text{in } \Omega \times [0, T], \quad (7.1)$$

$$v_t^2 + M^2 v^2 + N^2 v^2 + G^2 \nabla q^2 = 0, \quad \text{in } \Omega \times [0, T]. \quad (7.2)$$

Let $H^1 = (G^1)^{-1}$ and $H^2 = (G^2)^{-1}$. Multiplying (7.1) and (7.2) by H^1 and H^2 respectively, and denote $K = \max\{\|v^1\|_{L^\infty(0, T; H_s)}, \|v^2\|_{L^\infty(0, T; H_s)}\}$.

Subtracting the two equations and taking inner product in $L^2(\Omega)$ with $v^1 - v^2$, then one gets

$$\begin{aligned} 0 &= (H^1 v_t^1 - H^2 v_t^2, v^1 - v^2)_{L^2(\Omega)} + (\nabla q^1 - \nabla q^2, v^1 - v^2)_{L^2(\Omega)} \\ &\quad + (H^1(M^1 v^1 + N^1 v^1) - H^2(M^2 v^2 + N^2 v^2), v^1 - v^2)_{L^2(\Omega)} \\ &:= I_1 + I_2 + I_3, \end{aligned}$$

Denote $l_{v^1}, \omega_{v^1}, l_{v^2}, \omega_{v^2}$ by L^1, R^1, L^2, R^2 respectively.

$$\begin{aligned}
I_2 &= \int_{\Omega} \nabla(q^1 - q^2) \cdot (v^1 - v^2) dy \\
&= \int_{\partial\Omega} (q^1 - q^2)(v^1 - v^2) \cdot \vec{n} d\sigma \\
&= \int_{\partial\Omega} (q^1 - q^2)(L^1 - L^2) \cdot \vec{n} d\sigma + \int_{\partial\Omega} (q^1 - q^2)(R^1 - R^2) \times y \cdot \vec{n} d\sigma \\
&= m(L^1 - L^2)' \cdot (L^1 - L^2) + m(R^1 \times L^1 - R^2 \times L^2) \cdot (L^1 - L^2) \\
&\quad + \bar{J}(R^1 - R^2)' \cdot (R^1 - R^2) - (\bar{J}R^1 \times R^1 - \bar{J}R^2 \times R^2) \cdot (R^1 - R^2) \\
&= \frac{1}{2}m \frac{d}{dt} |L^1 - L^2|^2 + \frac{1}{2} \frac{d}{dt} [(\bar{J}(R^1 - R^2)) \cdot (R^1 - R^2)] \\
&\quad + m(R^1 \times L^1 - R^2 \times L^2) \cdot (L^1 - L^2) - (\bar{J}R^1 \times R^1 - \bar{J}R^2 \times R^2) \cdot (R^1 - R^2).
\end{aligned} \tag{7.3}$$

The term I_1 can be estimated as follows,

$$\begin{aligned}
I_1 &= (H^1 v_t^1 - H^2 v_t^2, v^1 - v^2)_{L^2(\Omega)} \\
&= (H^1(v^1 - v^2)_t, v^1 - v^2)_{L^2(\Omega)} + ((H^1 - H^2)v_t^2, v^1 - v^2)_{L^2(\Omega)} \\
&:= I_{11} + I_{12}.
\end{aligned} \tag{7.4}$$

From the definition of G ,

$$\begin{aligned}
I_{11} &= (H^1(v^1 - v^2)_t, v^1 - v^2)_{L^2(\Omega)} \\
&= (J_{X^1}^T J_{X^1}(v^1 - v^2)_t, v^1 - v^2)_{L^2(\Omega)} \\
&= (J_{X^1}(v^1 - v^2)_t, J_{X^1}(v^1 - v^2))_{L^2(\Omega)} \\
&= \frac{1}{2} \frac{d}{dt} (J_{X^1}(v^1 - v^2), J_{X^1}(v^1 - v^2))_{L^2(\Omega)} \\
&\quad - \left(\frac{\partial J_{X^1}}{\partial t}(v^1 - v^2), J_{X^1}(v^1 - v^2) \right)_{L^2(\Omega)}.
\end{aligned} \tag{7.5}$$

Therefore,

$$I_{11} \geq \frac{1}{2} \frac{d}{dt} \|J_{X^1}(v^1 - v^2)\|_{L^2(\Omega)}^2 - C(T, K) \sup_{s \in [0, t]} \|v^1(s) - v^2(s)\|_{L^2(\Omega)}^2. \tag{7.6}$$

On the other hand, by Lemma 3.5,

$$\begin{aligned}
|I_{12}| &\leq C \|g_{ij}^1 - g_{ij}^2\|_{L^\infty(\Omega)} \|v^1 - v^2\|_{L^2(\Omega)} \|v_t^2\|_{L^2(\Omega)} \\
&\leq C(T, K) \left(\sup_{s \in [0, t]} (|l^1(s) - l^2(s)| + |\omega^1(s) - \omega^2(s)|) \right) \sup_{s \in [0, t]} \|v^1(s) - v^2(s)\|_{L^2(\Omega)} \\
&\leq C \sup_{s \in [0, t]} \|v^1(s) - v^2(s)\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned} \tag{7.7}$$

I_3 can be estimated similarly,

$$|I_3| \leq C \sup_{[0,t]} \|v^1(s) - v^2(s)\|_{L^2(\mathbb{R}^3)}^2.$$

Therefore,

$$\begin{aligned} & \frac{d}{dt} \|J_{X^1}(v^1 - v^2)\|_{L^2(\Omega)}^2 + m \frac{d}{dt} |L^1 - L^2|^2 + \frac{d}{dt} [(\bar{J}(R^1 - R^2)) \cdot (R^1 - R^2)] \\ & \leq C \sup_{[0,t]} \|v^1(s) - v^2(s)\|_{L^2(\mathbb{R}^3)}^2. \end{aligned}$$

It follows that

$$\begin{aligned} & \|J_{X^1}(v^1 - v^2)(t)\|_{L^2(\Omega)}^2 + m |L_1 - L_2|^2 + (\bar{J}(R_1 - R_2)) \cdot (R_1 - R_2) \\ & \leq C \int_0^t \sup_{s \in [0,\tau]} \|v^1(s) - v^2(s)\|_{L^2(\mathbb{R}^3)}^2 d\tau \end{aligned}$$

Since X^1 is always a diffeomorphism and \bar{J} is positive definite, thus

$$\|v^1 - v^2\|_{L^2(\mathbb{R}^3)}^2(t) \leq C \int_0^t \sup_{\tau \in [0,s]} \|v^1 - v^2\|_{L^2(\mathbb{R}^3)}^2(\tau) ds \quad (7.8)$$

By the Gronwall's inequality, $v^1 = v^2$ *a.e.* in $[0, T] \times \mathbb{R}^3$. Uniqueness, as in [18] or earlier publication [24] for the Euler equations, combining the fact that the system is reversible, implies

$$v \in C([0, T]; H_s) \cap C^1([0, T]; H_{s-1}).$$

In fact, from the preceding estimates, one can easily get that

$$\frac{d}{dt} \|v\|_{H^s(\Omega)}^2 + m \frac{d}{dt} |L|^2 + \frac{d}{dt} [\bar{J}R \cdot R] \leq C \|v\|_{H^s}^2 (\|\nabla v\|_{L^\infty(\Omega)} + \|v\|_{L^2(\mathbb{R}^3)} + 1). \quad (7.9)$$

It implies that once $\|\nabla v\|_{L^\infty(\Omega)}$ does not blow up, $\|v\|_{H^s}$ will not blow up. Using the argument as in the paper [18], one can get that the lifespan of the solution does not depend on s .

Acknowledgements

Gratitude is expressed specially to supervisor Professor Zhouping Xin and the referees for their careful reading of the manuscript and their fruitful suggestions.

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